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The *bc*-system of higher rank revisited

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Abstract

The *bc*-system of higher rank introduced recently is examined further. It is shown that the correlation functions are connected with certain non-Abelian θ-functions and it is discussed how quasi-determinants arise.

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1. Introduction

The usual *bc*-system living on Riemann surfaces of arbitrary genus is an important ingredient in bosonic string theory [1, 2] and has been treated in a completely rigorous way by Raina in [3, 4]. Assuming some natural physical axioms, Raina showed the existence and uniqueness of the correlation functions and was able to rederive the explicit expressions (involving theta-functions) using the geometry of the theta-divisor. It is important to note that one considers in this approach not the quantum fields b, c themselves (which should be 'operator-valued sections' of certain line bundles), but their correlation functions inheriting the symmetries of the operators. A closely related cousin of the *bc*-sytem based on a Hermitian vector bundle of rank r was introduced in [5] and the existence and uniqueness of the correlation functions was established for a particular class of bundles. Using a result of [6], it was shown in [7] that the determinants of the correlation functions are sections of pullbacks of generalized theta-line bundles. Since the *bc*-system of higher rank is free, one expects that the higher correlation functions (e.g. the four-point function) are given as some kind of determinants of propagators, i.e. two-point functions. This 'generalized Wick's theorem' seems to be out of reach at the moment, but there does exist a relation between certain determinants of the correlation functions. The underlying 'addition theorem' for non-Abelian theta-functions was found by Fay [8] and was reformulated recently by Polishchuk [9] with the help of quasi-determinants introduced by Gel'fand and Retakh [10, 11]. Unfortunately, these results are too weak to yield a description of the correlation functions of the associated 'non-Abelian $U(1)$-current' (obtained by the regularization processes from the field correlation functions); for a discussion of the current see [7, 12]. Nevertheless, the beautiful connection between the physical model and the mathematics associated with vector bundles on Riemann surfaces deserves further study. We strongly believe that a thorough understanding of the bc-system of higher rank will shed light on a rigorous (algebro-geometric) free-field representation of Wess–Zumino–Witten models.

The paper is structured as follows. After a brief review of the bc-system of rank one in section 2, we describe the bc-system of higher rank in section 3. Here we establish a connection between the correlation functions and the above-mentioned quasi-determinants. We also derive an identity which reduces in the rank one case to the theorem of Wick. In section 4 some conclusions are presented. For the convenience of the reader we have collected some basic facts about quasi-determinants in the appendix.

2. The bc-system of rank one

In the following Σ_g will be a compact Riemann surface of genus $g \geqslant 2$ with canonical bundle $K \equiv K_{\Sigma_g}$. The set of (isomorphism classes of) holomorphic line bundles of degree d will be denoted by $Pic^d(\Sigma_g)$. The canonical theta-divisor is defined by

$$\Theta := \left\{ L \in Pic^{g-1}(\Sigma_g) \mid h^0(\Sigma_g, L) \neq 0 \right\} \subset Pic^{g-1}(\Sigma_g). \tag{1}$$

We will denote the inverse of the line bundle α by α^{-1}.

Let $\alpha \in Pic^{g-1}(\Sigma_g) \backslash \Theta$, i.e. α is a line bundle of degree $g-1$ and satisfies $h^0(\Sigma_g, \alpha) = 0$. In the associated (chiral) bc-system—given by an action $S \sim \int b\bar{\partial}c$—the field c (respectively b) is a section of α (respectively $K \otimes \alpha^{-1}$); note that there will be neither zero modes of b nor c due to our assumption on α. The bc-system with these choices is thus a system of (twisted) chiral fermions! The propagator $\langle b(z)c(w) \rangle$ is then a meromorphic section of the line bundle $(K \otimes \alpha^{-1}) \boxtimes \alpha := \pi_1^*(K \otimes \alpha^{-1}) \otimes \pi_2^*(\alpha)$ over $\Sigma_g \times \Sigma_g$ having a simple pole on the diagonal $\Delta \subset \Sigma_g \times \Sigma_g$; here $\pi_i \colon \Sigma_g \times \Sigma_g \to \Sigma_g$ for $i = 1, 2$, is the canonical projection onto the ith factor. Using the map $\phi_\alpha \colon \Sigma_g \times \Sigma_g \to Pic^{g-1}(\Sigma_g)$, given by

$$\phi_\alpha(z, w) := \mathcal{O}(z - w) \otimes \alpha \tag{2}$$

we pull back the θ line bundle from $Pic^{g-1}(\Sigma_g)$ to obtain [3, 4]

$$\phi_\alpha^*(\mathcal{O}(\Theta)) \simeq (K \otimes \alpha^{-1}) \boxtimes \alpha \otimes \mathcal{O}(\Delta). \tag{3}$$

Thus, the propagator is the meromorphic section of $\phi_\alpha^*(\mathcal{O}(\Theta)) \otimes \mathcal{O}(-\Delta)$. Since the normalized section of $\phi_\alpha^*(\mathcal{O}(\Theta))$ is given by the (uniquely determined) θ-function with characteristic α, i.e. by $\frac{\vartheta[\alpha](z-w)}{\vartheta[\alpha](0)}$, and the normalized section of $\mathcal{O}(\Delta)$ is given by the prime form $E(z, w)$, we get the result

$$\langle b(z)c(w) \rangle = \frac{\vartheta[\alpha](z - w)}{\vartheta[\alpha](0)E(z, w)} \equiv S_\alpha(z, w). \tag{4}$$

In the case that α is an even theta characteristic, i.e. $\alpha^{\otimes 2} = K$, the Szegö-kernel S_α—consequently the propagator also—is antisymmetric in its arguments.

In an analogous fashion one has maps $\phi_\alpha^n \colon (\Sigma_g \times \Sigma_g)^n \to Pic^{g-1}(\Sigma_g)$, given by

$$\phi_\alpha^n(z_1, w_1, \ldots, z_n, w_n) := \mathcal{O}\left(\sum_{i=1}^n z_i - \sum_{i=1}^n w_i \right) \otimes \alpha; \tag{5}$$

note that $\phi_\alpha^1 \equiv \phi_\alpha$ from above. The pullback of $\mathcal{O}(\Theta)$ under ϕ_α^n is given by [3, 4]

$$(\phi_\alpha^n)^*(\mathcal{O}(\Theta)) \simeq (K \otimes \alpha^{-1}) \boxtimes \alpha \boxtimes \cdots \boxtimes (K \otimes \alpha^{-1}) \boxtimes \alpha \otimes \mathcal{O}(D_n), \tag{6}$$

where D_n is the divisor of poles and zeros

$$D_n := \sum_{1 \leqslant i < j \leqslant 2n} (-1)^{i+j+1} \Delta_{ij} \tag{7}$$

and Δ_{ij} is the divisor where the ith and jth coordinates coincide. Hence, the $2n$-point function $\langle b(z_1)\cdots c(w_n)\rangle$ is given as the normalized section of $\left(\phi_\alpha^n\right)^*(\mathcal{O}(\Theta))\otimes\mathcal{O}(-D_n)$. Since the first factor leads to a theta-function and the second to a product of prime forms, we obtain an explicit expression for the $2n$-point functions. Comparing the expression for the 4-point function with the determinant of propagators (the *bc*-system is free, so the two expressions should coincide according to Wick's theorem), we obtain

$$\frac{\vartheta[\alpha](z_1-w_1+z_2-w_2)E(z_1,z_2)E(w_2,w_1)}{\vartheta[\alpha](0)E(z_1,w_1)E(z_1,w_2)E(z_2,w_1)E(z_2,w_2)}=\det\left(\frac{\vartheta[\alpha](z_i-w_j)}{\vartheta[\alpha](0)E(z_i,w_j)}\right) \tag{8}$$

which is equivalent to the trisecant identity of Fay [3, 13]. The analogous comparison of the higher correlation functions with the corresponding determinant of propagators yields the general Fay identity.

3. The *bc*-system of higher rank

Let us denote by $\mathcal{U}_{\Sigma_g}(r,d)$ the moduli space of (isomorphism classes of) stable vector bundles of rank r and degree d on Σ_g; recall that a bundle F on a Riemann surface is called stable if $\mu(G)<\mu(F)$ for every proper subbundle $G\subset F$ (where $\mu(F):=\frac{d}{r}$ for a bundle F of rank r and degree d). In the particular case $d=r(g-1)$ the non-Abelian theta-divisor is defined in close analogy to (1) by

$$\Theta_r:=\left\{F\in\mathcal{U}_{\Sigma_g}(r,r(g-1))\mid h^0(\Sigma_g,F)\neq 0\right\}\subset\mathcal{U}_{\Sigma_g}(r,r(g-1)).$$

We will denote the dual bundle of F by $F^\vee$ and the highest nonvanishing exterior power of a vector bundle F by $\det(F)$.

Recall [5, 7] that the *bc*-system of higher rank (which we will call the bc_r-system in the following) associated with a holomorphic (Hermitian) vector bundle E of rank r is defined by the action

$$S\sim\int_{\Sigma_g}b\bar{\partial}_E c;$$

here the field c (resp. b) is a section of E (resp. $K\otimes E^\vee$) and $\bar{\partial}_E$ is the Dolbeault-operator acting on the smooth sections of E; in the case $r=1$ we denote the corresponding line bundle by α. As mentioned in the introduction, the $2n$-point functions $\langle b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle$ exist and are uniquely determined if we choose E from the complement of the non-Abelian theta-divisor, i.e. if $E\in\mathcal{U}_{\Sigma_g}(r,r(g-1))\setminus\Theta_r$. Let us denote by Δ one of the 2^{2g} theta-characteristics satisfying $\Delta^{\otimes 2}\simeq K$ (since the diagonal appears only in the form $\mathcal{O}(\Delta)$, there should be no confusion about the meaning of Δ). Tensoring with Δ may be regarded as a map $\Delta\otimes - : \mathcal{U}_{\Sigma_g}(r,0)\to\mathcal{U}_{\Sigma_g}(r,r(g-1))$, so that we can pull back the non-Abelian theta-divisor:

$$\Delta^*\Theta_r:=\left\{\chi\in\mathcal{U}_{\Sigma_g}(r,0)\mid h^0(\Sigma_g,\Delta\otimes\chi))>0\right\}\subset\mathcal{U}_{\Sigma_g}(r,0). \tag{9}$$

Consequently, if $\chi\in\mathcal{U}_{\Sigma_g}(r,0)\setminus\Delta^*\Theta_r$, then $E\equiv\Delta\otimes\chi\in\mathcal{U}_{\Sigma_g}(r,r(g-1))\setminus\Theta_r$ is of the type considered in the bc_r-system. Thus, the field c (resp. b) is a section of $\Delta\otimes\chi$ (resp. $\Delta\otimes\chi^\vee$), i.e. a χ-valued (resp. $\chi^\vee$-valued) spinor and we interpret the bc_r-system in this case as a system of chiral fermions of rank r.

The $2n$-point functions are sections of the vector bundle $p_1^*(K\otimes E^\vee)\otimes p_2^*(E)\otimes\cdots\otimes p_{2n-1}^*(K\otimes E^\vee)\otimes p_{2n}^*(E)$ over $\Sigma_g^{2n}:=\Sigma_g\times\cdots\times\Sigma_g$ ($2n$ times); here $p_i\colon\Sigma_g^{2n}\to\Sigma_g$ are the canonical projections onto the ith factor. In particular, the 2-point function $\langle b(z)c(w)\rangle$ is given by the non-Abelian Szegö-kernel $S_E(z,w)$ of Fay [5, 8],

$$\langle b(z)c(w)\rangle=S_E(z,w). \tag{10}$$

It is given locally by an $r \times r$-matrix and has an expansion

$$S_E(z, w) = \frac{I_r}{z - w} + a_0(w; E) + a_1(w; E)(z - w) + \cdots, \tag{11}$$

where the coefficients $a_i(\cdot; E)$ are described in [8]. Unfortunately, one has only very little explicit information about them. For the higher correlation functions the situation is even worse. Here one has neither a direct representation (in contrast to the rank one case) nor a 'generalized Wick's theorem' which would allow a representation of the higher correlation functions as certain 'determinants' of propagators.

Let us consider the geometric aspects of the bc_r-system following [7]. For $E \in \mathcal{U}_{\Sigma_g}(r, r(g-1)) \backslash \Theta_r$ there is a map

$$\phi_E : \Sigma_g \times \Sigma_g \to \mathcal{U}_{\Sigma_g}(r, r(g-1)) \tag{12}$$

defined in close analogy to (2) by

$$\phi_E(z, w) := \mathcal{O}(z - w) \otimes E.$$

The pullback of the non-Abelian theta-line bundle $\mathcal{O}(\Theta_r)$ is given in complete analogy to (3) by [6]:

$$\phi_E^*(\mathcal{O}(\Theta_r)) \simeq (K^{\otimes r} \otimes \det(E^\vee)) \boxtimes \det(E) \otimes \mathcal{O}(\Delta)^{\otimes r}. \tag{13}$$

Hence, the determinant of the propagator is the meromorphic section of the line bundle $\phi_E^*(\mathcal{O}(\Theta_r)) \otimes \mathcal{O}(-\Delta)^{\otimes r}$. In the rank one case we could use the fact that a section of $\phi_\alpha^*(\mathcal{O}(\Theta))$ is given by a theta-function to obtain explicit expressions. The one-dimensionality of the space of theta-functions (of level one) is expressed by $h^0\big(Pic^{g-1}(\Sigma_g), \mathcal{O}(\Theta)\big) = 1$. It is a fundamental result of [14] (and was essential for the mathematical proofs of the Verlinde formula) that this can be generalized to 'non-Abelian theta-functions', i.e. we have

$$h^0\left(\mathcal{U}_{\Sigma_g}(r, r(g-1)), \mathcal{O}(\Theta_r)\right) = 1.$$

Let us denote the uniquely determined holomorphic section of $\mathcal{O}(\Theta_r)$ by $\vartheta_r(\cdot)$. Then $\vartheta_r \circ \phi_E$ is a holomophic section of $\phi_E^*(\mathcal{O}(\Theta_r))$ over $\Sigma_g \times \Sigma_g$ and one has

$$\vartheta_r \circ \phi_E(z, w) = \vartheta_r(\phi_E(z, w)) = \vartheta_r(E \otimes \mathcal{O}(z - w)).$$

To obtain a close analogy to the rank one case, we define the corresponding non-Abelian theta-function ('with characteristic E') by

$$\vartheta_r[E](z - w) := \vartheta_r(E \otimes \mathcal{O}(z - w)) \tag{14}$$

and call it the non-Abelian theta-function associated with $E \in \mathcal{U}_{\Sigma_g}(r, r(g-1)) \backslash \Theta_r$. Note that this is a rather formal definition, since one has at the moment no explicit formulae for these non-Abelian theta-functions. Let us nevertheless proceed. The determinant of the propagator is the meromorphic section of $\phi_E^*(\mathcal{O}(\Theta_r)) \otimes \mathcal{O}(-\Delta)^{\otimes r}$. Since the uniquely determined section of $\mathcal{O}(\Delta)$ is the prime form $E(z, w)$, we may thus write

$$\det(\langle b(z)c(w)\rangle) = \frac{\vartheta_r[E](z - w)}{\vartheta_r[E](0)E(z, w)^r}. \tag{15}$$

Let us now consider the $2n$-point function $\langle b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle$, which is a meromorphic section of the vector bundle $(K \otimes E^\vee) \boxtimes E \boxtimes \cdots \boxtimes (K \otimes E^\vee) \boxtimes E$ having simple poles (resp. zeros) whenever two arguments of different (resp. same) types coincide. In close analogy to the rank one case considered in (5) we define a map $\phi_E^n : (\Sigma_g \times \Sigma_g)^n \longrightarrow \mathcal{U}_{\Sigma_g}(r, r(g-1))$ by setting

$$\phi_E^n(z_1, w_1, \ldots, z_n, w_n) := \mathcal{O}\left(\sum_{i=1}^n z_i - \sum_{i=1}^n w_i\right) \otimes E;$$

note that again $\phi_E^1 \equiv \phi_E$. According to [6], the pullback of the non-Abelian theta-line bundle is then given in analogy to (6) by

$$\begin{aligned}\left(\phi_E^n\right)^* (\mathcal{O}(\Theta_r)) \simeq (K^{\otimes r} \otimes \det(E^\vee)) \boxtimes \det(E) \boxtimes \cdots \\ \cdots \boxtimes (K^{\otimes r} \otimes \det(E^\vee)) \boxtimes \det(E) \otimes \mathcal{O}(D_n)^{\otimes r}\end{aligned}$$

where D_n is the divisor defined in (7) and there are n factors of $(K^{\otimes r} \otimes \det(E^\vee)) \boxtimes \det(E)$. Note that this reduces to (13) for $n = 1$ since $D_1 = \Delta_{12} \equiv \Delta$. Thus, the (appropriately interpreted) determinant of the $2n$-point function is a section of $\left(\phi_E^n\right)^*(\mathcal{O}(\Theta_r)) \otimes \mathcal{O}(-D_n)^{\otimes r}$. Generalizing the construction from above, $\vartheta_r \circ \phi_E^n$ is the holomorphic section of $\left(\phi_E^n\right)^*(\mathcal{O}(\Theta_r))$ on $(\Sigma_g \times \Sigma_g)^n$. Thus,

$$\vartheta_r \circ \phi_E^n(z_1, w_1, \ldots, z_n, w_n) = \vartheta_r\left(\phi_E^n(z_1, \ldots, w_n)\right) = \vartheta_r\left(E \otimes \mathcal{O}\left(\sum_{i=1}^n z_i - \sum_{i=1}^n w_i\right)\right).$$

In close analogy to the case $n = 1$ of (14) we write

$$\vartheta_r[E](z_1 - w_1 + \cdots + z_n - w_n) := \vartheta_r \circ \phi_E^n(z_1, w_1, \ldots, z_n, w_n)$$

and abbreviate this also by

$$\vartheta_r[E]\left(\sum_{i=1}^n (z_i - w_i)\right) \equiv \vartheta_r[E](z_1 - w_1 + \cdots + z_n - w_n).$$

Therefore, the determinant of the $2n$-point function $\langle b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle$ is given formally by

$$\det{}_n(\langle b(z_1)\cdots c(w_n)\rangle) = \frac{\vartheta_r[E]\left(\sum_{i=1}^n (z_i - w_i)\right)}{\vartheta_r[E](0)} \left(\frac{\prod_{1\leqslant i<j\leqslant n} E(z_i, z_j)E(w_j, w_i)}{\prod_{1\leqslant i,j\leqslant n} E(z_i, w_j)}\right)^r. \tag{16}$$

Here we have introduced a subscript to denote the difference from the usual determinant; note that $\det_1$ is the usual determinant, i.e. $\det_1 \equiv \det$. Before we reformulate these expressions, we return to the rank one case; we thus assume that $\alpha \in Pic^{g-1}(\Sigma_g)\backslash\Theta$. We also change the notation slightly (for a better comparison with [9, 8]) and label the points as $z_0, w_0, z_1, w_1, \ldots, z_n, w_n$; so, we consider the $2(n+1)$-point function of the *bc*-system associated with α. Comparing (2) and (5) we find

$$\begin{aligned}\phi_\alpha^2(z_0, w_0, z_1, w_1) &\equiv \alpha \otimes \mathcal{O}(z_0 - w_0 + z_1 - w_1) \\ &= \alpha(z_1 - w_1) \otimes \mathcal{O}(z_0 - w_0) \\ &\equiv \phi_{\alpha(z_1-w_1)}(z_0, w_0)\end{aligned} \tag{17}$$

where we have used the abbreviation $\alpha(z_1 - w_1) := \alpha \otimes \mathcal{O}(z_1 - w_1)$. This implies for the pullbacks of the theta-line bundle

$$\begin{aligned}\left(\phi_\alpha^2\right)^* (\mathcal{O}(\Theta))_{|(z_0,w_0,z_1,w_1)} &= \mathcal{O}(\Theta)_{|\phi_\alpha^2(z_0,w_0,z_1,w_1)} \\ &\overset{(17)}{=} \mathcal{O}(\Theta)_{|\phi_{\alpha(z_1-w_1)}(z_0,w_0)} \\ &= \phi^*_{\alpha(z_1-w_1)}(\mathcal{O}(\Theta))_{|(z_0,w_0)}.\end{aligned}$$

The corresponding holomorphic section of $\left(\phi_\alpha^2\right)^*(\mathcal{O}(\Theta))$ in (z_0, w_0, z_1, w_1) is given by $\vartheta[\alpha](z_0 - w_0 + z_1 - w_1)$, so that we may write

$$\vartheta[\alpha](z_0 - w_0 + z_1 - w_1) = \vartheta[\alpha(z_1 - w_1)](z_0 - w_0). \tag{18}$$

The right-hand side appears in the Szegö-kernel associated with $\alpha(z_1 - w_1)$:

$$S_{\alpha(z_1-w_1)}(z_0, w_0) \equiv \frac{\vartheta[\alpha(z_1 - w_1)](z_0 - w_0)}{\vartheta[\alpha(z_1 - w_1)](0)E(z_0, w_0)}. \tag{19}$$

Using (18) we can rewrite the left-hand side of (8) to obtain the following identity:

$$S_{\alpha(z_1-w_1)}(z_0, w_0)\frac{E(z_0, z_1)E(w_0, w_1)}{E(z_0, w_1)E(w_0, z_1)} = \frac{\det(S_\alpha(z_i, w_j))}{S_\alpha(z_1, w_1)}. \tag{20}$$

Let us introduce a concise notation. Define the 2×2-matrix

$$\mathbf{S}_\alpha(z_0, w_0; z_1, w_1) := \begin{pmatrix} S_\alpha(z_0, w_0) & S_\alpha(z_0, w_1) \\ S_\alpha(z_1, w_0) & S_\alpha(z_1, w_1) \end{pmatrix} \tag{21}$$

with commutative entries $S_\alpha(z_i, w_j)$. Its 00-quasi-determinant can be written according to (A3) as

$$|\mathbf{S}_\alpha(z_0, w_0; z_1, w_1)|_{00} = \frac{\det \mathbf{S}_\alpha(z_0, w_0; z_1, w_1)}{\det S_\alpha(z_1, w_1)} \equiv \frac{\det(S_\alpha(z_i, w_j))}{S_\alpha(z_1, w_1)}.$$

Combining this with (20), we have shown the following proposition.

Proposition 1. *Let $\alpha \in Pic^{g-1}(\Sigma_g)\backslash\Theta$. Wick's theorem holds for the bc-system associated with α if and only if the following identity is satisfied:*

$$S_{\alpha(z_1-w_1)}(z_0, w_0)\frac{E(z_0, z_1)E(w_0, w_1)}{E(z_0, w_1)E(w_0, z_1)} = |\mathbf{S}_\alpha(z_0, w_0; z_1, w_1)|_{00}. \tag{22}$$

This identity is equivalent to the trisecant identity of Fay.

As mentioned in the introduction, the equivalence of this identity to the trisecant identity of Fay is due to Polishchuk [9], and the equivalence of Wick's theorem to the trisecant identity is due to Raina [3, 4] (this connection was noticed before, see, e.g. [15–17]). The general identity is obtained as follows. Let $\{z_0, z_1, \ldots, z_n\}$ and $\{w_0, w_1, \ldots, w_n\}$ be disjoint sets of points on Σ_g and consider the bundle

$$\alpha\left(\sum_{i=1}^n (z_i - w_i)\right) := \alpha \otimes \mathcal{O}\left(\sum_{i=1}^n (z_i - w_i)\right).$$

If the points have been chosen in such a way that $h^0\left(\Sigma_g, \alpha\left(\sum_{i=1}^n (z_i - w_i)\right)\right) = 0$, then the associated Szegö-kernel $S_{\alpha(\sum_{i=1}^n (z_i - w_i))}(z_0, w_0)$ (generalizing the expression (19)) exists and can be written with the help of (18) as some kind of $2(n+1)$-point function. The corresponding $(n+1) \times (n+1)$-matrix of propagators is defined by

$$\mathbf{S}_\alpha(z_0, w_0; \{z_i\}, \{w_i\}) := \begin{pmatrix} S_\alpha(z_0, w_0) & \cdots & S_\alpha(z_0, w_n) \\ \vdots & & \vdots \\ S_\alpha(z_n, w_0) & \cdots & S_\alpha(z_n, w_n) \end{pmatrix} \tag{23}$$

and the identity generalizing (22) is

$$S_{\alpha(\sum_{i=1}^n (z_i - w_i))}(z_0, w_0) \prod_{i=1}^n \frac{E(z_0, z_i)E(w_0, w_i)}{E(z_0, w_i)E(w_0, z_i)} = |\mathbf{S}_\alpha(z_0, w_0; \{z_i\}, \{w_i\})|_{00} \tag{24}$$

where the quasi-determinant of a matrix of higher rank is defined inductively in (A1).

Now, we come back to the case of higher rank. As mentioned before, the identity (24) was generalized by Fay (see $(2.16)'$ in [8]) to the case of vector bundles of higher rank and was formulated with the help of quasi-determinants by Polishchuk [9]. More precisely, one has the following theorem.

Theorem 2 (Fay, Polishchuk)**.** *Let* $E \in \mathcal{U}_{\Sigma_g}(r, r(g-1)) \setminus \Theta_r$ *and let* $\{z_0, z_1, \ldots, z_n\}$ *and* $\{w_0, w_1, \ldots, w_n\}$ *be disjoint sets of points on* Σ_g *such that* $E\big(\sum_{i=1}^n (z_i - w_i)\big) \in \mathcal{U}_{\Sigma_g}(r, r(g-1)) \setminus \Theta_r$ *(i.e.* $h^0\big(\Sigma_g, E\big(\sum_{i=1}^n (z_i - w_i)\big)\big) = 0$*). Let* $\mathbf{S}_E(z_0, w_0; \{z_i\}, \{w_i\})$ *be the* $(n+1) \times (n+1)$*-matrix analogous to (23) but this time with noncommuting entries* $S_E(z_i, w_j)$*. Then the following identity holds:*

$$S_{E(\sum_{i=1}^n (z_i - w_i))}(z_0, w_0) \prod_{i=1}^n \frac{E(z_0, z_i)E(w_0, w_i)}{E(z_0, w_i)E(w_0, z_i)} = |\mathbf{S}_E(z_0, w_0; \{z_i\}, \{w_i\})|_{00}. \tag{25}$$

This identity reduces in the rank one case to the identity (24) which is equivalent to the trisecant identity of Fay, so, in this sense, (25) represents a ‘matrix-valued trisecant identity’. It should be stressed that the entries $S_E(z_i, w_j)$ are $r \times r$-matrices. For other approaches to addition formulae for non-Abelian theta-functions see [6, 18].

Taking the determinant of the right-hand side of (25) yields

$$\det(|\mathbf{S}_E(z_0, w_0; \{z_i\}, \{w_i\})|_{00}). \tag{26}$$

On the other hand, taking the determinant of the left-hand side of (25) yields

$$\det\left(S_{E(\sum_{i=1}^n (z_i - w_i))}(z_0, w_0)\right) \left(\prod_{i=1}^n \frac{E(z_0, z_i)E(w_0, w_i)}{E(z_0, w_i)E(w_0, z_i)}\right)^r. \tag{27}$$

Combining (10) and (15), one finds that the determinant of the non-Abelian Szegö-kernel is given by a non-Abelian theta-function. In the case at hand one obtains the analogon to (19), i.e.

$$\det\left(S_{E(\sum_{i=1}^n (z_i - w_i))}(z_0, w_0)\right) = \frac{\vartheta_r\left[E\left(\sum_{i=1}^n (z_i - w_i)\right)\right](z_0 - w_0)}{\vartheta_r\left[E\left(\sum_{i=1}^n (z_i - w_i)\right)\right](0) E(z_0, w_0)^r}.$$

Since the maps ϕ_E^n are defined in a completely analogous way to the maps ϕ_α^n of the rank one case, one consequently obtains an equation analogous to (17) and finally to (18), i.e.

$$\vartheta_r\left[E\left(\sum_{i=1}^n (z_i - w_i)\right)\right](z_0 - w_0) = \vartheta_r[E]\left(\sum_{i=0}^n (z_i - w_i)\right).$$

Therefore, we have found for the determinant of the left-hand side of (25) the expression

$$\frac{\vartheta_r[E]\left(\sum_{i=0}^n (z_i - w_i)\right)}{\vartheta_r[E]\left(\sum_{i=1}^n (z_i - w_i)\right) E(z_0, w_0)^r} \left(\prod_{i=1}^n \frac{E(z_0, z_i)E(w_0, w_i)}{E(z_0, w_i)E(w_0, z_i)}\right)^r.$$

Using (16) we may now express the non-Abelian theta-functions through the determinants of the correlation functions of the bc_r-system associated with E. The prime forms cancel and all that remains is

$$\frac{\det_{n+1}(\langle b(z_0)c(w_0)b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle)}{\det_n(\langle b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle)}. \tag{28}$$

Thus, by comparing (26) and (28), we show the following proposition.

Proposition 3. *Under the assumptions of theorem 2 one obtains by taking the determinants on both sides of (25) the following identity for the correlation functions of the* bc_r*-system associated with E:*

$$\frac{\det_{n+1}(\langle b(z_0)c(w_0)b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle)}{\det_n(\langle b(z_1)c(w_1)\cdots b(z_n)c(w_n)\rangle)} = \det(|\mathbf{S}_E(z_0, w_0; \{z_i\}, \{w_i\})|_{00}). \tag{29}$$

In the rank one case there are no determinants and the quasi-determinant on the right-hand side is—according to (A3)—the quotient of the determinant of the entire $(n+1)\times(n+1)$-matrix $\mathbf{S}_\alpha(z_0, w_0; \{z_i\}, \{w_i\})$ (which is in this case the $2(n+1)$-point function $\langle b(z_0)\cdots c(w_n)\rangle$ due to Wick's theorem) and the determinant of the $n\times n$-matrix $\mathbf{S}_\alpha(z_1, w_1; \{z_2,\ldots,z_n\}, \{w_2,\ldots,w_n\})$ (which is the $2n$-point function $\langle b(z_1)\cdots c(w_n)\rangle$). This verifies the identity (29) in the rank one case.

The identity (29) may be used to express the $2(n+1)$-point function through a $2n$-point function, which may be expressed through a $2(n-1)$-point function, and so on. Thus, we may proceed inductively and express the higher correlation function as a kind of product of determinants of quasi-determinants. Let us introduce a convenient abbreviation for the $(n+1-j)\times(n+1-j)$-matrices appearing in the following consideration:

$$\mathbf{S}_E(z_j, w_j; \mathbf{z}_j, \mathbf{w}_j) := \mathbf{S}_E(z_j, w_j; \{z_{j+1},\ldots,z_n\}, \{w_{j+1},\ldots,w_n\}). \tag{30}$$

Note that these matrices are exactly the $(n+1-j)\times(n+1-j)$-matrices in the lower right corner of the $(n+1)\times(n+1)$-matrix $\mathbf{S}_E(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$. With this notation we obtain

$$\begin{aligned}
&\det_{n+1}(\langle b(z_0)\cdots c(w_n)\rangle)\\
&\qquad = \det\{|\mathbf{S}_E(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)|_{00}\}\det_n(\langle b(z_1)\cdots c(w_n)\rangle)\\
&\qquad = \det\{|\mathbf{S}_E(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)|_{00}\}\det\{|\mathbf{S}_E(z_1, w_1; \mathbf{z}_1, \mathbf{w}_1)|_{11}\}\\
&\qquad\quad \times \det_{n-1}(\langle b(z_2)\cdots c(w_n)\rangle)\\
&\qquad\ \vdots\\
&\qquad = \left(\prod_{k=0}^{n-1}\det\{|\mathbf{S}_E(z_k, w_k; \mathbf{z}_k, \mathbf{w}_k)|_{kk}\}\right)\det(\langle b(z_n)c(w_n)\rangle).
\end{aligned}$$

Since $\langle b(z_n)c(w_n)\rangle = S_E(z_n, w_n)$ and since the quasi-determinant of a 1×1-matrix is just the corresponding entry, it makes sense to define for the last factor

$$\det\{|\mathbf{S}_E(z_n, w_n; \mathbf{z}_n, \mathbf{w}_n)|_{nn}\} := \det\{S_E(z_n, w_n)\} \equiv \det(\langle b(z_n)c(w_n)\rangle). \tag{31}$$

Collecting the above results, we may formulate the following theorem.

Theorem 4. *Let the assumptions be as in theorem 2. Using the notation (30) and the convention (31), the determinants of the correlation functions of the bc_r-system associated with E can be represented as*

$$\det_{n+1}(\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle) = \prod_{k=0}^{n}\det\{|\mathbf{S}_E(z_k, w_k; \mathbf{z}_k, \mathbf{w}_k)|_{kk}\}. \tag{32}$$

In the rank one case, i.e. for $\alpha \in Pic^{g-1}(\Sigma_g)\setminus\Theta$, this identity reduces to the usual Wick theorem:

$$\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle = \det \mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0).$$

Proof. The first equation has already been shown; the reduction in the rank one case remains to be shown. Since in this case there are no determinants, (32) reduces to

$$\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle = \prod_{k=0}^{n}|\mathbf{S}_\alpha(z_k, w_k; \mathbf{z}_k, \mathbf{w}_k)|_{kk}.$$

Note that $\mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$ is the $(n+1)\times(n+1)$-matrix we started from; let us abbreviate it as $T = (t_{ij})$. The factor belonging to $k = 0$ is therefore $|T|_{00}$. The matrix $\mathbf{S}_\alpha(z_1, w_1; \mathbf{z}_1, \mathbf{w}_1)$ arises from $\mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$ by deleting the zeroth row and the zeroth column, so that the factor belonging to $k = 1$ is given by $|T^{0,0}|_{11}$. The next matrix $\mathbf{S}_\alpha(z_2, w_2; \mathbf{z}_2, \mathbf{w}_2)$ arises from

$\mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$ by deleting the first two rows and columns, so that the factor belonging to $k = 2$ is given by $|T^{01,01}|_{22}$. So, the right-hand side is given by

$$|T|_{00}|T^{0,0}|_{11}|T^{01,01}|_{22}\cdots|T^{012\cdots n-2,012\cdots n-2}|_{n-1n-1}t_{nn}.$$

According to [10, 11], this is precisely the usual determinant of T! Thus, substituting back $T \equiv \mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$, one obtains the identity

$$\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle = \det \mathbf{S}_\alpha(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0),$$

which is exactly the usual form of Wick's theorem. □

Note that we always 'expanded' the matrices in the upper-left corner. Of course, we could have expanded the large matrix in completely different ways. Let us consider first the case $n = 1$ and let us start with the rank one case. Abbreviate $A = \mathbf{S}_\alpha(z_0, w_0; z_1, w_1)$, so that $A = (a_{ij})_{1\leqslant i,j\leqslant 2}$ with $a_{ij} = S_\alpha(z_{i-1}, w_{j-1})$. Due to (A3) one has for arbitrary $1 \leqslant i, j, k, l \leqslant 2$ the identity $(-1)^{i+j}|A|_{ij}\det A^{ij} = (-1)^{k+l}|A|_{kl}\det A^{kl}$, coming from the different expansions of the original matrix A. Since we always have in our application $i = j$ and $k = l$, essentially one relation remains, namely $|A|_{11}a_{22} = |A|_{22}a_{11}$, i.e.

$$\frac{|\mathbf{S}_\alpha(z_0, w_0; z_1, w_1)|_{00}}{S_\alpha(z_0, w_0)} = \frac{|\mathbf{S}_\alpha(z_0, w_0; z_1, w_1)|_{11}}{S_\alpha(z_1, w_1)}.$$

This identity comes in a more physical interpretation from the two different expansions of the 4-point function of the bc-system associated with α. We expect that a similar relation holds in higher rank, provided one takes the appropriate determinants. An exchange $z_0 \leftrightarrow z_1$ and $w_0 \leftrightarrow w_1$ yields instead of (25) for $n = 1$ the identity

$$S_{E(z_0-w_0)}(z_1, w_1)\frac{E(z_1, z_0)E(w_1, w_0)}{E(z_1, w_0)E(w_1, z_0)} = |\mathbf{S}_E(z_0, w_0; z_1, w_1)|_{11}.$$

Taking the determinant gives as above

$$\frac{\det_2(\langle b(z_0)c(w_0)b(z_1)c(w_1)\rangle)}{\det(\langle b(z_0)c(w_0)\rangle)} = \det\{|\mathbf{S}_E(z_0, w_0; z_1, w_1)|_{11}\}.$$

This is a second representation for the determinant of the 4-point function. A comparison with the case $n = 1$ of (29) shows that indeed

$$\frac{\det\{|\mathbf{S}_E(z_0, w_0; z_1, w_1)|_{00}\}}{\det S_E(z_0, w_0)} = \frac{\det\{|\mathbf{S}_E(z_0, w_0; z_1, w_1)|_{11}\}}{\det S_E(z_1, w_1)}. \tag{33}$$

In the general case we abbreviate as above $T \equiv \mathbf{S}_E(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$. The right-hand side of (32) can then be written as (see the proof of theorem 4)

$$\det\{|T|_{00}\}\det\left\{|T^{0,0}|_{11}\right\}\det\left\{|T^{01,01}|_{22}\right\}\cdots\det\left\{|T^{01\cdots n-1,01\cdots n-1}|_{nn}\right\}$$

where we have used the obvious convention for the last factor. Therefore, (32) can be written in the form

$$\det_{n+1}(\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle) = \prod_{k=0}^{n}\det\left\{|T^{0\cdots k-1,0\cdots k-1}|_{kk}\right\} \tag{34}$$

where the factor corresponding to $k = 0$ is just $\det\{|T|_{00}\}$. In terms of the original matrix T we have used the 0th, 1st, 2nd, ..., nth diagonal element to expand. Now, let $\sigma = \begin{pmatrix} 0 & 1 & \cdots & n \\ \sigma(0) & \sigma(1) & \cdots & \sigma(n)\end{pmatrix} \in S_{n+1}$ be an arbitrary permutation. Then we can also expand first along the $\sigma(0)$th element, then the $\sigma(1)$st element, and so on. Instead of (34) we then obtain the identity

$$\det_{n+1}(\langle b(z_0)c(w_0)\cdots b(z_n)c(w_n)\rangle) = \prod_{k=0}^{n}\det\left\{|T^{\sigma_k,\sigma_k}|_{\sigma(k)\sigma(k)}\right\} \tag{35}$$

where we have abbreviated $\sigma_k := \sigma(0)\cdots\sigma(k-1)$. Note that this reduces to (34) in the particular case $\sigma = \mathrm{id}$.

Proposition 5. *Let the assumptions be as in theorem 4 and let us abbreviate $T \equiv \mathbf{S}_E(z_0, w_0; \mathbf{z}_0, \mathbf{w}_0)$. For two arbitrary permutations $\sigma, \tau \in S_{n+1}$ one has the following identity (where again $\sigma_k := \sigma(0)\cdots\sigma(k-1)$ and similarly for τ):*

$$\prod_{k=0}^{n} \det\left\{|T^{\sigma_k,\sigma_k}|_{\sigma(k)\sigma(k)}\right\} = \prod_{k=0}^{n} \det\left\{|T^{\tau_k,\tau_k}|_{\tau(k)\tau(k)}\right\}. \tag{36}$$

Proof. According to (35), both sides are equal to the determinant of the $2(n+1)$-point function of the bc_r-system. □

In the case $n = 1$ there are only two elements in S_2: the identity $\sigma = \mathrm{id}$ and the transposition $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The identity (36) then reduces to (33).

4. Conclusion

In this paper the bc-system of higher rank was considered further, in particular its connection to the quasi-determinants of Gel'fand and Retakh. The determinants of the correlation functions were expressed through non-Abelian theta-functions, showing a geometrical connection to a particular Wess–Zumino–Witten (WZW) model (the non-Abelian theta-functions are closely related to the spaces of conformal blocks of WZW models, see, e.g., [19]). Although a 'generalized Wick's theorem' is still lacking, one may regard theorem (4) as a very weak version of it (and, hopefully, as a first step in the right direction). If one had this identity one could calculate with its help the higher correlation functions of the current and the energy–momentum tensor, thus allowing a better comparison to WZW models; for the current of the bc-system of higher rank see [7, 12]. Recall that in the rank one case Wick's theorem is equivalent to the trisecant identity of Fay [3, 4] which one may view as the mathematical counterpart to the Abelian bosonization [15–17]. We are convinced that a further study will show a very close connection between the sought for 'Wick's theorem of higher rank', addition laws of non-Abelian theta-functions and WZW models (i.e. non-Abelian bosonization [20]).

Appendix

Here we have collected some basic facts about quasi-determinants which were introduced by Gel'fand and Retakh [10, 11]. Let $A = (a_{ij})$ with $1 \leqslant i, j \leqslant n$ be a $n \times n$-matrix with formal noncommutative entries. For $1 \leqslant p, q \leqslant n$ we define inductively n^2 quasi-determinants $|A|_{pq}$ which are rational in the entries a_{ij}. In the case $n = 1$ we simply set $|A|_{11} := a_{11}$. Denote the $(n-1) \times (n-1)$-matrix which results by deleting the kth row and the lth column from A by A^{kl}. Then we define

$$|A|_{pq} := a_{pq} - \sum_{\substack{i \neq p \\ j \neq q}} a_{pj} |A^{pq}|_{ij}^{-1} a_{iq}. \tag{A1}$$

Let us consider as an example a 2×2-matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The four quasi-determinants are

$$\begin{aligned} |A|_{11} &= a_{11} - a_{12}a_{22}^{-1}a_{21} \qquad & |A|_{12} &= a_{12} - a_{11}a_{21}^{-1}a_{22} \\ |A|_{21} &= a_{21} - a_{22}a_{12}^{-1}a_{11} \qquad & |A|_{22} &= a_{22} - a_{21}a_{11}^{-1}a_{12}. \end{aligned} \tag{A2}$$

In the case that all variables are commutative, one obtains the following connection to the usual determinant:

$$|A|_{pq} = (-1)^{p+q}\frac{\det A}{\det A^{pq}}. \tag{A3}$$

This may be checked easily by hand in the case of a 2×2-matrix using (A2). If we delete the k_1th and k_2th rows and the l_1th and l_2th columns from A, we will denote the resulting $(n-2) \times (n-2)$-matrix by $A^{k_1k_2,l_1l_2}$. More generally, if we delete the k_1th, $\ldots$, k_rth rows and the l_1th, $\ldots$, l_rth columns of A, the resulting $(n-r) \times (n-r)$-matrix will be denoted by $A^{k_1\cdots k_r,l_1\cdots l_r}$.

As stressed in [10, 11], these quasi-determinants have certain homological properties (which the usual determinant has not). In particular, if the a_{ij}: $V_j \to V_i$ are invertible morphisms in an additive category allowing rational functions in the morphisms of objects, the quasi-determinant $|A|_{pq}$ is a morphism from V_q to V_p; this may be checked immediately in the case $n = 2$ using (A2). If one does not require that all quasi-deteminants exist at the same time, one may allow that some of the entries a_{ij} are not invertible; e.g. if one is interested only in $|A|_{11}$, then only a_{22} has to be invertible. Considering as objects V_j vector spaces of the dimensions d_j (so that the morphisms a_{ij} are $d_i \times d_j$ matrices), we may allow $d_i \neq d_j$. Much more information about quasi-determinants can be found in [10, 11].

References

[1] Green M, Schwarz J and Witten E 1987 *Superstring Theory I, II* (Cambridge: Cambridge University Press)

[2] D'Hoker E and Phong D H 1988 The geometry of string perturbation theory *Rev. Mod. Phys.* **60** 917–1065

[3] Raina A 1989 Fay's trisecant identity and conformal field theory *Commun. Math. Phys.* **122** 625–41

[4] Raina A 1990 Analyticity and chiral fermions on a Riemann surface *Helv. Phys. Acta* **63** 694–704

[5] Schork M 2000 Generalized *bc*-systems based on Hermitian vector bundles *J. Math. Phys.* **41** 2443–59

[6] Gómez Gonzáles E and Plaza Martín F J 2000 Addition formula for non-Abelian theta functions *Preprint* math.AG/0008005

[7] Schork M 2001 On the correlation functions of the vector bundle generalization of the *bc*-system *J. Math. Phys.* **42** 4563–9

[8] Fay J 1992 Kernel functions, analytic torsion, and moduli spaces *Mem. AMS* **464**

[9] Polishchuk A 2001 Triple Massey products on curves, Fay's trisecant identity and tangents to the canonical embedding *Preprint* math.AG/0107194

[10] Gel'fand I M and Retakh V S 1991 Determinants of matrices over noncommutative rings *Funct. Anal. Appl.* **25** 91–102

[11] Gel'fand I M and Retakh V S 1992 A theory of noncommutative determinants and characteristic functions of graphs *Funct. Anal. Appl.* **26** 231–46

[12] Schork M 2002 On the current of chiral fermions of higher rank *Lett. Math. Phys.* submitted

[13] Fay J 1973 *Theta Functions on Riemann Surfaces* (Berlin-New York: Springer)

[14] Beauville A, Narasimhan M S and Ramanan S 1989 Spectral covers and the generalized theta divisor *J. reine angew. Math.* **398** 169–79

[15] Eguchi T and Ooguri H 1987 Chiral bosonization on a Riemann surface *Phys. Lett.* B **187** 127–34

[16] Verlinde E and Verlinde H 1987 Chiral bosonization, determinants and the string partition function *Nucl. Phys.* B **288** 357–96

[17] Alvarez-Gaumé L, Bost J B, Moore G, Nelson Ph and Vafa C 1987 Bosonization on higher genus Riemann surfaces *Commun. Math. Phys.* **112** 503–52

[18] Ball J A and Vinnikov V 1999 Zero-pole interpolation for matrix meromorphic functions on a compact Riemann surface and a matrix Fay trisecant identity *Am. J. Math.* **121** 841–88

[19] Beauville A and Laszlo Y 1994 Conformal blocks and generalized theta functions *Commun. Math. Phys.* **164** 385–419

[20] Witten E 1984 Non-Abelian bosonization in two dimensions *Commun. Math. Phys.* **92** 455–72